Question. Consider a Markov chain with

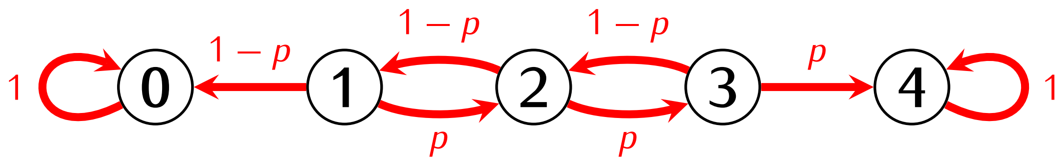
- states $S_1, \dots, S_N$
- a transition matrix P
- state vectors $X_0, X_1, \dots$

What can we say about X_n when n is large?

Example. The weather model:

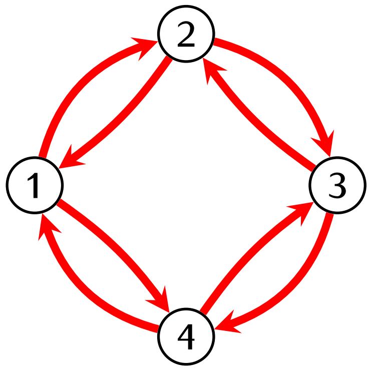
$$P = \begin{matrix} & \begin{matrix} R & S \end{matrix} \\ \begin{matrix} R \\ S \end{matrix} & \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \end{matrix}$$

Example. The gambling model:



$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccc}
1 & 1-p & 0 & 0 & 0 \\
0 & 0 & 1-p & 0 & 0 \\
0 & p & 0 & 1-p & 0 \\
0 & 0 & p & 0 & 0 \\
0 & 0 & 0 & p & 1
\end{array}\right]
\end{array}
$$

Example. Random walk on a circular network:



$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

The steady-state vector

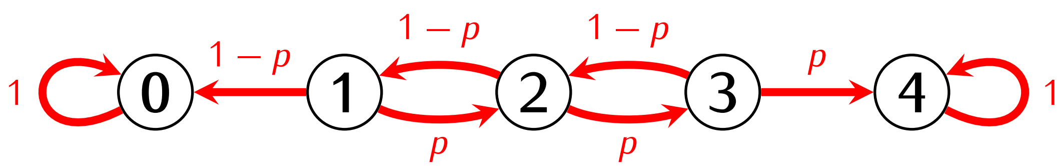
Definition

If P is a stochastic matrix then the *steady-state vector* of P is a probability vector Y such that $PY = Y$.

Example. The weather model:

$$P = \begin{array}{c} \\ \text{R} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{cc} \text{R} & \text{S} \end{array} \\ \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \end{array}$$

Example. The gambling model (with $p \neq 0, 1$):



$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 1 & 1-p & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & p & 1 \end{array}\right] \end{array}$$

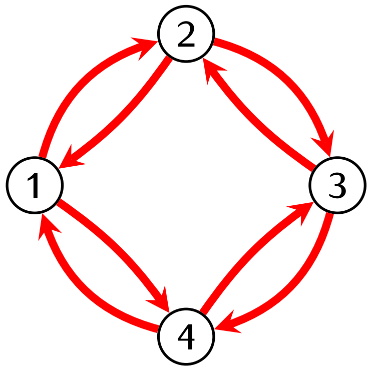
Proposition

If P is a stochastic matrix then P has a steady-state vector.

Lemma

If A is a square matrix then λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

Example. Random walk on a circular network:



$$P = \begin{array}{c} \\ \color{red}{1} \\ \color{red}{2} \\ \color{red}{3} \\ \color{red}{4} \end{array} \begin{array}{c} \begin{array}{cccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} \end{array} \\ \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

Definition

A stochastic matrix P is *regular* if there is $N \geq 0$ such that all entries of P^N are positive.

Perron-Frobenius Theorem

If P is a regular stochastic matrix then:

- There exists only one steady state vector Y of P
- For any probability vector X we have

$$\lim_n P^n X = Y$$