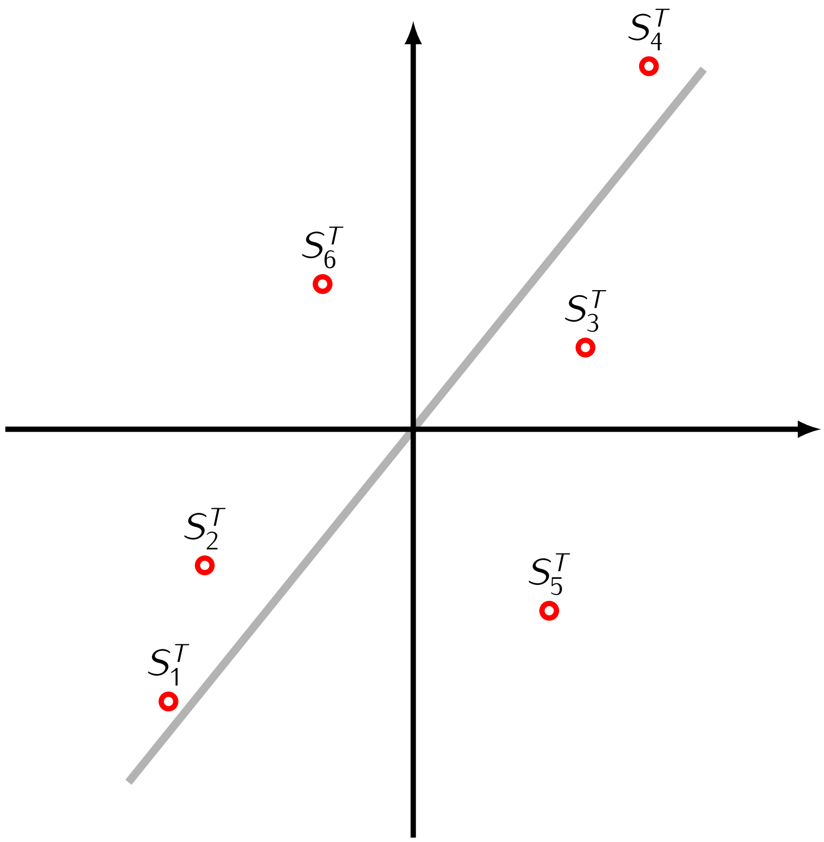
Example.

Demeaned data matrix:

$$A = \begin{array}{r} \\ \text{Aly} \\ \text{Bob} \\ \text{Chen} \\ \text{Deb} \\ \text{Emma} \\ \text{Finn} \end{array} \begin{array}{c} \begin{array}{cc} \text{Ex 1} & \text{Ex 2} \end{array} \\ \begin{bmatrix} -27 & -30 \\ -23 & -15 \\ 19 & 6 \\ 26 & 40 \\ 15 & -20 \\ -10 & 16 \end{bmatrix} \end{array} = \begin{bmatrix} \boxed{X_1} & \boxed{X_2} \end{bmatrix} = \begin{bmatrix} \boxed{S_1} \\ \boxed{S_2} \\ \boxed{S_3} \\ \boxed{S_4} \\ \boxed{S_5} \\ \boxed{S_6} \end{bmatrix}$$

Let $\mathbf{u}_1$ be the 1st principal axis of A, and let Y_1 be the 1st principal component of A:

$$Y_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} S_1\mathbf{u}_1 \\ S_2\mathbf{u}_1 \\ \vdots \\ S_N\mathbf{u}_1 \end{bmatrix}$$



The projection matrix

The difference matrix

Definition

Let A an $N \times M$ demeaned data matrix

$$A = \begin{bmatrix} \boxed{\quad S_1 \quad} \\ \boxed{\quad S_2 \quad} \\ \vdots \\ \boxed{\quad S_N \quad} \end{bmatrix}$$

Let $\mathbf{u}_1$ be the 1st principal axis of A, and let Y_1 be the 1st principal component of A.

The *2nd principal axis* of A is the 1st principal axis of the difference matrix

$$D_1 = A - Y_1 \mathbf{u}_1^T$$

The *2nd principal component* Y_2 of A is the 1st principal component of the matrix D_1.

Computation of the 2nd principal axis of A

Proposition

Given a demeaned data matrix A, let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$

be eigenvalues of the covariance matrix C_A and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ be orthonormal vectors such that $\mathbf{u}_i$ is an eigenvector of C_A corresponding to the eigenvalue λ_i.

- The *2nd principal axis* of A is the vector $\mathbf{u}_2$.
- The *2nd principal component* of A is the vector $Y_2 = A\mathbf{u}_2$.
- We have $\text{Var}(Y_2) = \lambda_2$.
- In addition, $\text{Cov}(Y_1, Y_2) = 0$.

The i^{th} principal component

Proposition/Definition

Given a demeaned data matrix A, let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$

be eigenvalues of the covariance matrix C_A and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ be orthonormal vectors such that $\mathbf{u}_i$ is an eigenvector of C_A corresponding to the eigenvalue λ_i.

- The i^{th} *principal axis* of A is the vector $\mathbf{u}_i$.
- The i^{th} *principal component* of A is the vector $Y_i = A\mathbf{u}_i$.
- We have $\text{Var}(Y_i) = \lambda_i$.
- In addition, $\text{Cov}(Y_i, Y_j) = 0$ if $i \neq j$.