Definition

Let A be an $n \times n$ matrix. A *quadratic form* defined by A is the function

$$q_A\colon \mathbb{R}^n \to \mathbb{R}$$

given by $q_A(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Proposition

Let A be an $n \times n$ matrix, and let $A_S = \frac{1}{2}(A + A^T)$. Then:

1) A_S is a symmetric matrix.

2) $q_A(\mathbf{v}) = q_{A_S}(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^n$.

Example.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Upshot. When defining a quadratic form q_A we can always assume that the matrix A is symmetric.

Change of variables in a quadratic form

Recall: If A is an $n \times n$ symmetric matrix then

$$A = QDQ^T$$

where:

$$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \qquad \text{orthogonal matrix, } Q^TQ = I_n$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{u}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{u}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{u}_n \end{array}$$

Upshot. For any vector $\mathbf{v} \in \mathbb{R}^n$ we have

$$q_A(\mathbf{v}) = q_D(Q^T\mathbf{v})$$

$$q_D(\mathbf{v}) = q_A(Q\mathbf{v})$$

Application: Classification of quadratic forms

Definition

Let A be an $n \times n$ matrix. The quadratic form q_A is

- *positive definite* if $q_A(\mathsf{v}) > 0$ for all $\mathsf{v} \in \mathbb{R}^n - \{0\}$
- *positive semidefinite* if $q_A(\mathsf{v}) \geq 0$ for all $\mathsf{v} \in \mathbb{R}^n - \{0\}$
- *negative definite* if $q_A(\mathsf{v}) < 0$ for all $\mathsf{v} \in \mathbb{R}^n - \{0\}$
- *negative semidefinite* if $q_A(\mathsf{v}) \leq 0$ for all $\mathsf{v} \in \mathbb{R}^n - \{0\}$
- *indefinite* if q_A has both positive and negative values.

Lemma

If D is a diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Then q_D is:

- positive definite if $\lambda_i > 0$ for $i = 1, \dots, n$
- positive semidefinite if $\lambda_i \geq 0$ for $i = 1, \dots, n$
- negative definite if $\lambda_i < 0$ for $i = 1, \dots, n$
- negative semidefinite if $\lambda_i \leq 0$ for $i = 1, \dots, n$
- indefinite if $\lambda_i > 0$ and $\lambda_j < 0$ for some i, j.

Lemma

Let A be a symmetric matrix with an orthogonal diagonalization

$$A = QDQ^T$$

If the quadratic form q_D is positive definite (positive semidefinite etc.) then q_A has the same property.

Proposition

Let A be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ then the quadratic form q_A is

- positive definite if $\lambda_i > 0$ for $i = 1, \ldots, n$
- positive semidefinite if $\lambda_i \geq 0$ for $i = 1, \ldots, n$
- negative definite if $\lambda_i < 0$ for $i = 1, \ldots, n$
- negative semidefinite if $\lambda_i \leq 0$ for $i = 1, \ldots, n$
- indefinite if $\lambda_i > 0$ and $\lambda_j < 0$ for some i, j.

Example. Classify the quadratic form q_A defined by the matrix

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

Proposition

Let A be a symmetric matrix. The quadratic form q_A id positive semidefinite if and only if there exists a matrix B such that $A = B^T B$.

Constrained optimization of quadratic forms

Constrained Maximum Problem. Given a symmetric matrix A, find a vector $\mathbf{v} \in \mathbb{R}^n$ such that $||\mathbf{v}|| = 1$ and the value $q_A(\mathbf{v})$ is the largest possible.

Lemma

If D is a diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then the vector $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$ is a solution of the Constrained Maximum Problem. Also, $q_D(\mathbf{e}_!) = \lambda_1$

Proposition

Let A be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. If $\mathbf{u}_1$ is an eigenvector corresponding to λ_1 such that $||\mathbf{u}_1|| = 1$ then $\mathbf{u}_1$ is a solution of the Constrained Maximum Problem and $q_A(\mathbf{u}_1) = \lambda_1$.

Example.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$