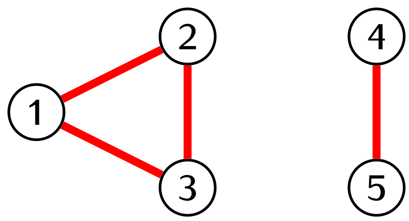
Note. From now on all graphs are simple, undirected unless it is indicated otherwise.

Definition

A graph is *connected* if any two vertices can be joined by a path.

A *connected component* of a graph is a maximal subgraph that is connected.



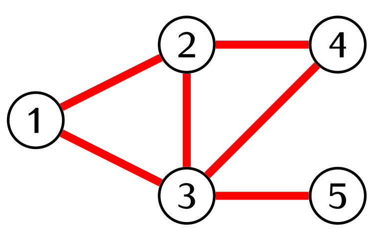
Goal:

- How to check if a graph is connected?
- If a graph is not connected, how to count its connected components?

Definition

If i is a vertex of a graph then the *degree* of i is the number

$$\deg(i) = (\text{the number of edges attached to } i)$$



Definition

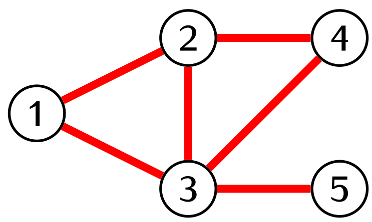
Let G be a graph with vertices $1, 2, \ldots, N$. The *Laplacian* of G is a matrix

$$L = D - A$$

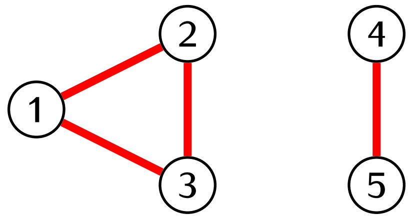
where

- A is the adjacency matrix of A
- D is a diagonal matrix with degrees of vertices on the diagonal.

Example.



Example.



Proposition $*$

If L is the Laplacian of a graph G then

$$\begin{pmatrix} \text{the number of} \\ \text{connected components} \\ \text{of } G \end{pmatrix} = \begin{pmatrix} \text{the number of} \\ \text{linearly independent eigenvectors} \\ \text{of } L \text{ corresponding to } \lambda = 0 \end{pmatrix}$$

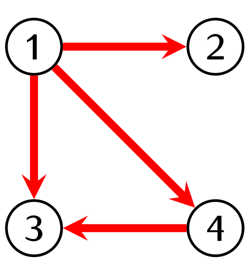
Definition

Let G be a directed graph with vertices $1, 2, \ldots, N$ and edges $e_1, e_2, \ldots, e_M$. The *edge incidence matrix* of G is an $N \times M$ matrix $B = (b_{ij})$ such that

- rows of B are labeled by vertices of G
- columns of B are labeled by edges of G
- the entries of B are given by

$$b_{ij} = \begin{cases} -1 & \text{if the edge } e_j \text{ starts at the vertex } i \\ +1 & \text{if the edge } e_j \text{ ends at the vertex } i \\ 0 & \text{otherwise} \end{cases}$$

Example.



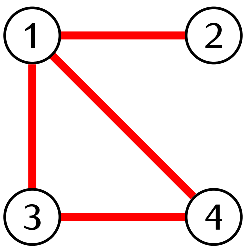
Lemma

Let

- G be a simple undirected graph
- L be the Laplacian of G
- B be the edge incidence matrix of G with the direction of edges selected in an arbitrary way.

Then $L = BB^T$.

Example.



Proof of Proposition $*$.

Proposition

If B is a any matrix then all eigenvalues of the matrix $A = BB^T$ are greater or equal to 0.

Corollary

If L is the Laplacian of a graph G then all eigenvalues of L are greater or equal to 0.