Recall: The simplex method works under the following assumptions:

1) The program is in the equality form: we want to maximize

$$z = c_1x_1 + \ldots + c_nx_n$$

subject to the constraints

$$\begin{array}{ccccc} a_{11}x_1 & + \ldots + & a_{1n}x_n & = & b_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1}x_1 & + \ldots + & a_{mn}x_n & = & b_m \end{array}$$

$$x_1,\ x_2,\ \ldots, x_n \geq 0$$

2) The coefficient matrix

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

is in the basic form.

3) $b_i \geq 0$ for $i = 1, \ldots, m$.

- These assumptions assure that the program has a basic feasible solution.
- **Phase I of the simplex method** finds some basic feasible solution or verifies that no feasible solutions exists.

Example. Maximize

$$z = x_1 + 2x_2$$

subject to:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 2 \\ -x_2 - x_3 &= -1 \end{aligned}$$

$$x_1,\ x_2, x_3 \geq 0$$

To get started we need some basic feasible solution

<u>Attempt 1</u>: Use x_1, x_2 as basic variables:

$$\begin{array}{c} \\ \\ (-1)\cdot \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1 & 2 \\ 0 & \boxed{-1} & -1 & -1 \end{array}\right] \end{array} \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] (-3) \to \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 0 & -2 & -1 \\ 0 & \boxed{1} & 1 & 1 \end{array}\right] \end{array}$$

<u>Solution:</u>

free | $x_3 = 0$

basic | $x_1 = -1$, $x_2 = 1$

Not feasible since $x_1 < 0$.

<u>Attempt 2</u>: Use x_1 and x_3 as free variables.

$$\begin{array}{c} \\ \\ (-1)\cdot \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1 & 2 \\ 0 & -1 & \boxed{-1} & -1 \end{array}\right] \end{array} \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] (-1) \to \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 3 & 0 & 1 \\ 0 & 1 & \boxed{1} & 1 \end{array}\right] \end{array}$$

<u>Solution:</u>

free | $x_3 = 0$

basic | $x_1 = 1$, $x_2 = 1$

This is a basic feasible solution, so we can use it to get the simplex method started.

Phase I of the simplex method

Assumption: The linear program is in the equality form with constraints

$$a_{i1}x_1 + \ldots + a_{in}x_n = b_i$$

for $i = 1, \ldots, m$.

1) Modify the constraints, if needed, so that $b_i \geq 0$ for $i = 1, \ldots, m$.

$$\begin{array}{l} x_1 + 3x_2 + x_3 = 2 \\ (-1)\cdot(-x_2 - x_3 = -1) \end{array} \quad \mapsto \quad \begin{array}{l} x_1 + 3x_2 + x_3 = 2 \\ x_2 + x_3 = 1 \end{array}$$

2) Add an additional variable s_i to the i-th constraint for $i = 1, \ldots, m$.

$$\begin{array}{l} x_1 + 3x_2 + x_3 = 2 \\ x_2 + x_3 = 1 \end{array} \quad \mapsto \quad \begin{array}{l} x_1 + 3x_2 + x_3 + s_1 = 2 \\ x_2 + x_3 + s_2 = 1 \end{array}$$

Note. The augmented matrix of the new constraints is in the basic form.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & s_1 & s_2 \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right]$$
$$\underbrace{x_1 \; x_2 \; x_3}_{\text{free}} \quad \underbrace{s_1 \; s_2}_{\text{basic}}$$

3) Use the simplex method to minimize the function

$$z = s_1 + \cdots + s_m$$

with the new constraints. If the minimum is non-zero, then the original linear program has not feasible solutions.

If the minimum is $z = 0$, then the solution that gives the minimum has $s_i = 0$ for $i = 1, \ldots, m$. In such case, values of the variables $x_1, \ldots, x_n$ give a basic feasible solution of the original linear program.