Example. Maximize

$$z = 3x_1 + x_2$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ 2x_1 + x_2 &\leq 7 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

Convert to the equality form:

Constraints:
$$\begin{aligned} -x_1 + x_2 + s_1 &= 1 \\ x_1 + s_2 &= 3 \\ 2x_1 + x_2 + s_3 &= 7 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

augmented $\longrightarrow$ matrix

$$\begin{array}{c} \overbrace{\qquad\quad x_1 \quad x_2}^{\text{free variables}} \; \overbrace{\quad s_1 \quad s_2 \quad s_3 \qquad}^{\text{basic variables}} \\ \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 3 \\ 2 & 1 & 0 & 0 & 1 & 7 \end{array}\right] \end{array}$$

We want to maximize

$$z = 3x_1 + x_2 + \underbrace{0s_1 + 0s_2 + 0s_3}_{\text{coefficients of basic variables are all } 0.}$$

Note: By setting the free variables to 0 we get a basic feasible solution:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ s_1 = 1 \\ s_2 = 3 \\ s_3 = 7 \end{cases} \qquad \text{this gives: } z = 0$$

Goal: Look for other basic feasible solutions that make the value of z larger.

Simplex tableau

$$\begin{array}{r|ccccc|c|l} & x_1 & x_2 & s_1 & s_2 & s_3 & & \\ \hline & -1 & 1 & 1 & 0 & 0 & 1 & s_1 \\ \text{constraints} & 1 & 0 & 0 & 1 & 0 & 3 & s_2 \\ & 2 & 1 & 0 & 0 & 1 & 7 & s_3 \\ \hline \text{the objective function} & 3 & 1 & 0 & 0 & 0 & z & \end{array}$$

The current basic feasible solution:

free $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$

basic $\begin{cases} s_1 = 1 \\ s_2 = 3 \\ s_3 = 7 \end{cases}$ $\mapsto \quad z = 0$

The pivot step:

1) We can increase the value of z by increasing either x_1 or x_2. Let's try to increase x_2, keep $x_1 = 0$.

2) Change of x_2 will affect the values of the basic variables s_1, s_2, s_3. These variables must stay ≥ 0, which restricts possible values of x_2.

Since by assumption $x_1 = 0$ we have:

$x_2 + s_1 = 1 \Rightarrow s_1 = 1 - x_2 \geq 0$ so: $x_2 \leq 1$

$0 \cdot x_2 + s_2 = 3 \Rightarrow s_2 = 3 \geq 0$ - no restriction on x_2

$x_2 + s_3 = 7 \Rightarrow s_3 = 7 - x_2 \geq 0$ so: $x_2 \leq 7$

All these conditions are satisfied if $x_2 \leq 1$.
For the biggest increase of z we set $x_2 = 1$ which makes $s_1 = 0$. Since we want free variables to have value 0, and basic variables to have non-zero values, we want to make x_2 into a basic variable and make s_1 free.

x_1	x_2	s_1	s_2	s_3	
-1	1	1	0	0	1
1	0	0	1	0	3
2	1	0	0	1	7
3	1	0	0	0	z

(-1) (-1)

$\longrightarrow$

free

x_1	x_2	s_1	s_2	s_3		
-1	1	1	0	0	1	x_2
1	0	0	1	0	3	s_2
3	0	-1	0	1	6	s_3
4	0	-1	0	0	$z-1$	

The new basic feasible solution

free: $x_1 = 0$, $s_1 = 0$

basic: $x_2 = 1$, $s_2 = 3$, $s_3 = 6$

$$\left\{\begin{array}{l} x_1 = 0 \\ s_1 = 0 \\ x_2 = 1 \\ s_2 = 3 \\ s_3 = 6 \end{array}\right. \longmapsto \quad z - 1 = 0 \quad (\text{or } z = 1)$$

The pivot step:

The free variables are x_1 and s_1.

Increasing s_1 will decrease z,

but increasing x_1 will increase z.

Thus we will increase x_1 while keeping $x_2, s_2, s_3 \geq 0$.

(Note: we keep $s_1 = 0$)

$-x_1 + x_2 = 1 \Rightarrow x_2 = 1 + x_1 \geq 0$ - no restriction on x_1

$x_1 + s_2 = 3 \Rightarrow s_2 = 3 - x_1 \geq 0$ so: $x_1 \leq 3$

$3x_1 + s_3 = 6 \Rightarrow s_3 = 6 - 3x_1 \geq 0$ so $3x_1 \leq 6$

$x_1 \leq 2$

The biggest value of x_1 satisfying all conditions is $x_1 = 2$

Then $s_3 = 0$, so s_3 becomes free and x_1 basic.

$\left(\frac{1}{3}\right)\cdot$	x_1	x_2	s_1	s_2	s_3	
	-1	1	1	0	0	1
	1	0	0	1	0	3
$\left(\frac{1}{3}\right)\cdot$	3	0	-1	0	1	6
	4	0	-1	0	0	$z-1$

$\longrightarrow$

	x_1	x_2	s_1	s_2	s_3	
	-1	1	1	0	0	1
	1	0	0	1	0	3
(1), (−1), (−4)	$\boxed{1}$	0	$-1/3$	0	$1/3$	2
	4	0	-1	0	0	$z-1$

The new basic feasible solution:

free

$$\begin{cases} s_1 = 0 \\ s_3 = 0 \\ x_2 = 3 \\ s_2 = 1 \\ x_1 = 2 \end{cases} \mapsto \quad z - 9 = 0 \quad (\text{or } z = 9)$$

basic

x_1	x_2	s_1 (free)	s_2	s_3 (free)		
0	1	$2/3$	0	$1/3$	3	x_2
0	0	$1/3$	1	$-1/3$	1	s_2
1	0	$-1/3$	0	$1/3$	2	x_1
0	0	$1/3$	0	$-4/3$	$z-9$	

The pivot step:

We can increase z by increasing s_1 (and keeping $s_3 = 0$)

We have:

$x_2 + \frac{2}{3}s_1 = 3 \Rightarrow x_2 = 3 - \frac{2}{3}s_1 \geq 0$ so: $\frac{2}{3}s_1 \leq 3$

$s_1 \leq \frac{9}{2}$

$\frac{1}{3}s_1 + s_2 = 1 \Rightarrow s_2 = 1 - \frac{1}{3}s_1 \geq 0$ so: $\frac{1}{3}s_1 \leq 1$

$s_1 \leq 3$

$x_1 - \frac{1}{3}s_1 = 2 \Rightarrow x_1 = 2 + \frac{1}{3}s_1 \geq 0$ no restriction on s_1

The largest value of s_1 satisfying all conditions is $s_1 = 3$. Then $s_2 = 0$, so s_2 becomes free and s_1 basic

	x_1	x_2	s_1	s_2	s_3	
	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	3
3·	0	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	1
	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2
	0	0	$\frac{1}{3}$	0	$-\frac{4}{3}$	$z-9$

$\longrightarrow$

	x_1	x_2	s_1	s_2	s_3	
	0	1	2/3	0	1/3	3
$(-\frac{2}{3})$ $(\frac{1}{3})$ $(-1/3)$	0	0	$\boxed{1}$	3	−1	3
	1	0	−1/3	0	1/3	2
	0	0	1/3	0	−4/3	$z-9$

$\downarrow$

x_1	x_2	s_1	s_2 (free)	s_3 (free)		
0	1	0	−2	1	1	x_2
0	0	1	3	−1	3	s_1
1	0	0	1	0	3	x_1
0	0	0	−1	−1	$z-10$	

The new basic feasible solution

free $\begin{cases} s_2 = 0 \\ s_3 = 0 \end{cases}$

basic $\begin{cases} x_2 = 1 \\ s_1 = 3 \\ x_1 = 3 \end{cases}$

$\longmapsto$ $z - 10 = 0$

(or: $z = 10$)

Note: The objective function depends on s_2 and s_3 only, but increasing these variables will make z smaller. Thus we can't increase z anymore, we reached the maximum.

The final solution of the linear program:

The maximum value is $z = 10$.

It is obtained for $x_1 = 3$, $x_2 = 1$

(and $s_1 = 3$, $s_2 = 0$, $s_3 = 0$)

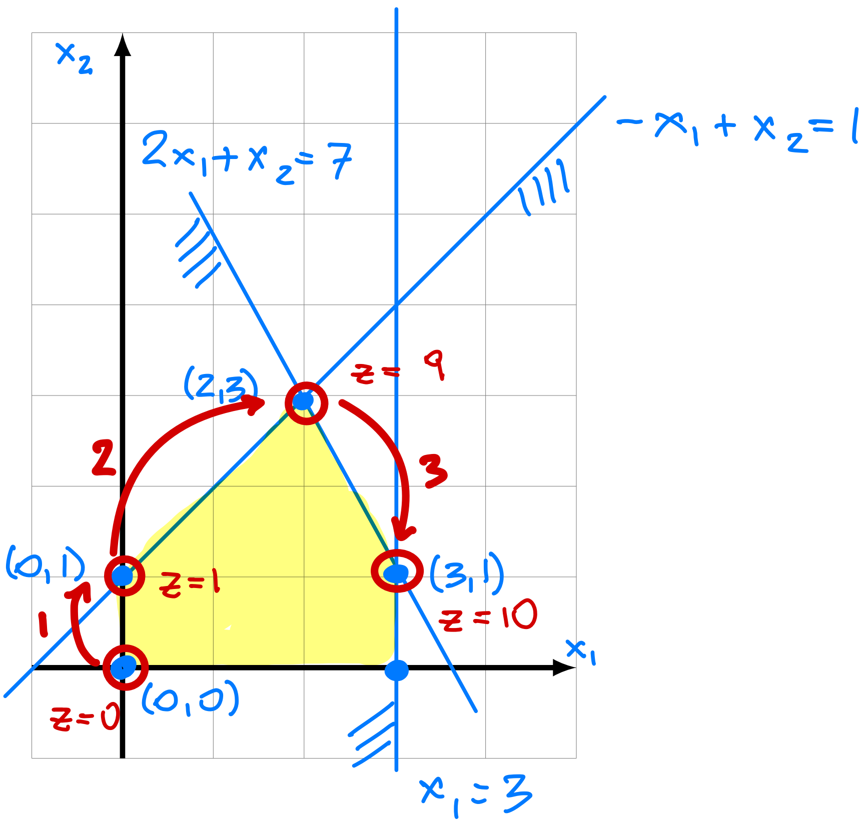
Geometric interpretation of the simplex method

Recall: Maximize

$$z = 3x_1 + x_2$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ 2x_1 + x_2 &\leq 7 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$



Note: It would have been more efficient to increase x_1 instead of x_2 in the first pivot step. In the simplex method there are various pivoting rules which aim to decide which free variable should be used at each step to get to the solution as quickly as possible.