The *simplex method* is one of the main methods of solving linear programs.

Special assumptions (for now):

1) The program is in the equality form: we want to maximize

$$z = c_1 x_1 + \ldots + c_n x_n$$

subject to the constraints:

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ \ldots \quad \ldots \quad \ldots \quad &\ldots \quad \ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$x_1,\ x_2,\ \ldots, x_n \geq 0$$

2) The coefficient matrix

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

is in the basic form.

3) $b_i \geq 0$ for $i = 1, \ldots, m$.

Example.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} -1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 & 1 & 7 \end{array}\right] \\ \underbrace{\qquad\qquad\qquad}_{\text{free}} \end{array} \longrightarrow \begin{cases} x_3 = 1 + x_1 - x_4 \\ x_2 = 3 - x_1 \\ x_5 = 7 - 2x_1 - x_4 \\ x_2 = \text{free} \\ x_4 = \text{free} \end{cases}$$

Basic feasible solutions

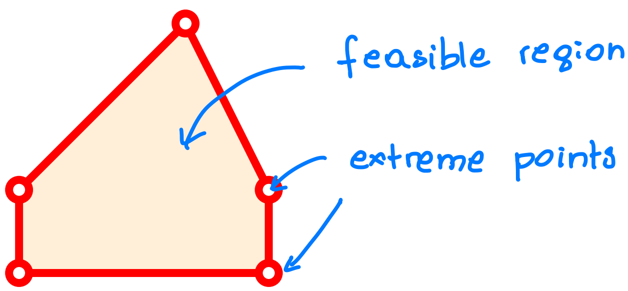
Basic feasible solutions are the solutions obtained by setting all free variables to 0.

Example.

$$\left[\begin{array}{ccccc|c} -1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 & 1 & 7 \end{array}\right] \qquad \begin{cases} x_3 = 1 + x_1 - x_4 \\ x_2 = 3 - x_1 \\ x_5 = 7 - 2x_1 - x_4 \\ x_1 = \text{free} \\ x_4 = \text{free} \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases} \xrightarrow[x_5 = 0]{x_1 = 0} \begin{cases} x_3 = 1 \\ x_2 = 3 \\ x_5 = 7 \\ x_1 = 0 \\ x_4 = 0 \end{cases}$$

a basic feasible solution

Geometric interpretation



Note: Basic feasible solutions correspond to extreme points of the feasible region.

The pivot step

$$(-1)\quad \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline -1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 & 1 & 7 \\ & & \uparrow & \uparrow & & \\ & & \text{basic} & \text{free} & & \end{array} \xrightarrow[\text{and } x_3 \text{ free}]{\text{make } x_4 \text{ basic}} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline -1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 3 & 0 & -1 & 0 & 1 & 6 \\ & & \uparrow & \uparrow & & \\ & & \text{free} & \text{basic} & & \end{array}$$

$$\begin{cases} x_4 = 1 + x_1 - x_3 \\ x_2 = 3 - x_1 \\ x_5 = 6 - 3x_1 + x_3 \\ x_1 = \text{free} \\ x_3 = \text{free} \end{cases}$$

a new basic feasible solution:

$$\begin{cases} x_4 = 1 \\ x_2 = 3 \\ x_5 = 6 \\ x_1 = 0 \\ x_3 = 0 \end{cases}$$

One more assumption

4) In the objective function $z = c_1x_1 + \ldots + c_nx_n$ the coefficients c_i corresponding to basic variables are equal to 0.

Example. The objective function:

$$4x_1 + \underbrace{2x_2}_{} + x_4 = z$$

non-zero coefficient of a basic variable

Constraints:

$$\begin{aligned} -x_1 + x_3 + x_4 &= 1 \\ x_1 + x_2 &= 3 \\ 2x_1 + x_4 + x_5 &= 7 \end{aligned}$$

x_1, x_4 - free

x_2, x_3, x_5 - basic

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0$$

Subtract side by side:

$$\begin{array}{l} 4x_1 + 2x_2 + x_4 = z \\ -2(x_1 + x_2 = 3) \\ \hline 2x_1 + \underbrace{0x_2}_{} + x_4 = z - 6 \end{array}$$

zero coefficient of the basic variable