Question. Consider a Markov chain with

- states $S_1, \ldots, S_N$
- a transition matrix P
- state vectors $X_0, X_1, \ldots$ $X_n = PX_{n-1} = P^n X_0$

What can we say about X_n when n is large?

Example. The weather model:

$$P = \begin{array}{c c} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \end{array}$$

$X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{array}{c} R \\ S \end{array}$ for n large: $X_n \approx \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}\begin{array}{c} R \\ S \end{array}$

$X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ —— " —— $X_n \approx \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}\begin{array}{c} R \\ S \end{array}$

$X_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ —— " —— $X_n \approx \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}\begin{array}{c} R \\ S \end{array}$

<u>Note:</u>

$$P^n \approx \begin{array}{c c} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{bmatrix} \end{array}$$ for large n

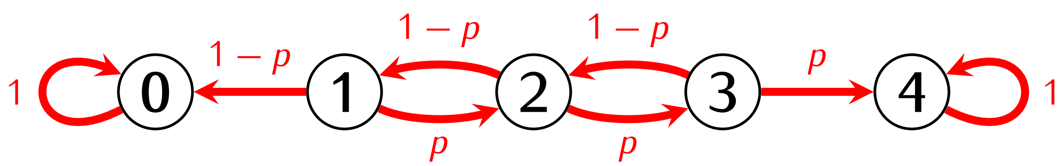
(converges, all colums the same)

For any $X_0 = \begin{bmatrix} p \\ q \end{bmatrix}$ we have:

$p + q = 1$

$$X_n = P^n X_0 \approx \begin{bmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} p + \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} q = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

Example. The gambling model:



$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
1 & 1-p & 0 & 0 & 0 \\
0 & 0 & 1-p & 0 & 0 \\
0 & p & 0 & 1-p & 0 \\
0 & 0 & p & 0 & 0 \\
0 & 0 & 0 & p & 1
\end{bmatrix}
\end{array}
$$

For $p = 0.5$:

$$
P^n \approx \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
1 & 0.75 & 0.5 & 0.25 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0.25 & 0.5 & 0.25 & 1
\end{bmatrix}
\end{array}
$$

(converges, different columns)

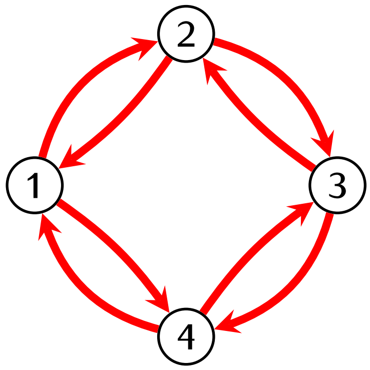
Note:

This means that $X_n = P^n X_0$ will depend on the choice of the vector X_0:

$$
X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow X_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
X_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow X_n = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}
$$

Example. Random walk on a circular network:



$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 1/2 & 0 & 1/2 \\ 4 & 1/2 & 0 & 1/2 & 0 \end{array}$$

A random walker can return to the starting position after an even number of steps only. Thus P^n will have 0 on the main diagonal if n is odd, and non-zero numbers if n is even.

Upshot: In this case P^n does not converge.

The steady-state vector

Assume that a Markov chain that starts with some state vector X_0 converges to some vector:

$$\lim_{n\to\infty} X_n = Y$$

We have: $X_n = P^n X_0$

So: $\lim_{n\to\infty} P^n X_0 = Y$

$$\lim_{n\to\infty} P(P^{n-1}X_0) = Y$$

$$\| $$

$$P\Big(\underbrace{\lim_{n\to\infty} P^{n-1}X_0}_{Y}\Big) = Y$$

We get: $PY = Y$

Definition

If P is a stochastic matrix then the *steady-state vector* of P is a probability vector Y such that $PY = Y$.

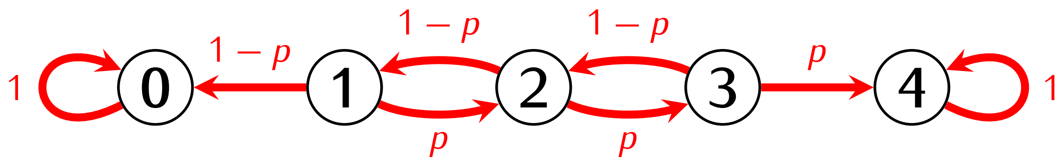
Note: Equivalently: Y is an eigenvector of P corresponding to the eigenvalue 1

Upshot: If P is the transition matrix of some Markov chain $X_0, X_1, \ldots$, and $\lim_{n\to\infty} X_n = Y$ then Y is a steady state vector of P.

Example. The weather model:

$$P = \begin{array}{c c} & \begin{array}{cc} \text{R} & \text{S} \end{array} \\ \begin{array}{c} \text{R} \\ \text{S} \end{array} & \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \end{array}$$

Example. The gambling model (with $p \neq 0, 1$):



$$P = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1-p & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & p & 1 \end{bmatrix} \end{array}$$

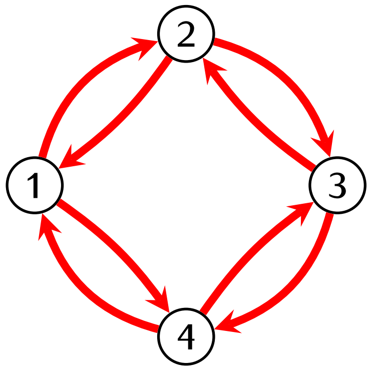
Proposition

If P is a stochastic matrix then P has a steady-state vector.

Lemma

If A is a square matrix then λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

Example. Random walk on a circular network:



$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

Definition

A stochastic matrix P is *regular* if there is $N \geq 0$ such that all entries of P^N are positive.

Perron-Frobenius Theorem

If P is a regular stochastic matrix then:

- There exists only one steady state vector Y of P
- For any probability vector X we have

$$\lim_n P^n X = Y$$