Definition

Let A be an $n \times n$ matrix. A *quadratic form* defined by A is the function

$$q_A \colon \mathbb{R}^n \to \mathbb{R}$$

given by $q_A(\mathbf{v}) = \mathbf{v}^T A \mathbf{v}$.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$q_A \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$q_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = [x_1 \ \ x_2] \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \ \ x_2] \cdot \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

$$= x_1(x_1 + 2x_2) + x_2(3x_1 + 4x_2)$$

$$= x_1^2 + 2x_1x_2 + 3x_1x_2 + 4x_2^2$$

$$= x_1^2 + 5x_1x_2 + 4x_2^2$$

Proposition

Let A be an $n \times n$ matrix, and let $A_S = \frac{1}{2}(A + A^T)$. Then:

1) A_S is a symmetric matrix.

2) $q_A(\mathbf{v}) = q_{A_S}(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^n$.

Proof:

1) $A_S^T = \left(\frac{1}{2}(A+A^T)\right)^T = \frac{1}{2}(A+A^T)^T = \frac{1}{2}(A^T + A) = A_S$

2) $$q_{A_S}(v) = v^T A_S v = v^T\left(\tfrac{1}{2}A + \tfrac{1}{2}A^T\right)v$$
$$= \tfrac{1}{2}v^TAv + \tfrac{1}{2}v^TA^Tv$$
$$= \tfrac{1}{2}v^TAv + \tfrac{1}{2}(v^TA^Tv)^T$$
$$= \tfrac{1}{2}v^TAv + \tfrac{1}{2}v^TAv$$
$$= v^TAv = q_A(v)$$

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad q_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^2 + 5x_1x_2 + x_2^2$$

$$A_S = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix}$$

$$q_{A_S}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = [x_1 \ x_2]\begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ x_2]\cdot\begin{bmatrix} x_1 + 5/2\, x_2 \\ 5/2\, x_1 + 4x_2 \end{bmatrix}$$
$$= x_1(x_1 + 5/2\, x_2) + x_2(5/2\, x_1 + 4x_2)$$
$$= x_1^2 + \tfrac{5}{2}x_1x_2 + \tfrac{5}{2}x_1x_2 + 4x_2^2 = x_1^2 + 5x_1x_2 + 4x_2^2$$
$$= q_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

Upshot. When defining a quadratic form q_A we can always assume that the matrix A is symmetric.

Change of variables in a quadratic form

Recall: If A is an $n \times n$ symmetric matrix then

$$A = QDQ^T$$

where:

$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ orthogonal matrix, $Q^TQ = I_n$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

λ_1 = eigenvalue corresponding to $\mathbf{u}_1$
λ_2 = eigenvalue corresponding to $\mathbf{u}_2$
... ...
λ_n = eigenvalue corresponding to $\mathbf{u}_n$

Upshot. For any vector $\mathbf{v} \in \mathbb{R}^n$ we have

$$q_A(\mathbf{v}) = q_D(Q^T\mathbf{v})$$

$$q_D(\mathbf{v}) = q_A(Q\mathbf{v})$$

$$\begin{tikzcd} v \arrow[r, mapsto] & Q^T v \\ \mathbb{R}^n \arrow[r] \arrow[dr, "q_A"'] & \mathbb{R}^n \arrow[d, "q_D"] \\ & \mathbb{R} \end{tikzcd}$$

$q_A(v) = v^TAv$

$$q_D(Q^Tv) = (Q^Tv)^T D (Q^Tv) = \underbrace{v^TQDQ^T}_{A}v = v^TAv$$

Note:

$$q_D\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = [x_1 \dots x_n] \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \dots x_n] \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix}$$

$$= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2$$

Application: Classification of quadratic forms

Definition

Let A be an $n \times n$ matrix. The quadratic form q_A is

- *positive definite* if $q_A(\mathbf{v}) > 0$ for all $\mathbf{v} \in \mathbb{R}^n - \{0\}$
- *positive semidefinite* if $q_A(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n - \{0\}$
- *negative definite* if $q_A(\mathbf{v}) < 0$ for all $\mathbf{v} \in \mathbb{R}^n - \{0\}$
- *negative semidefinite* if $q_A(\mathbf{v}) \leq 0$ for all $\mathbf{v} \in \mathbb{R}^n - \{0\}$
- *indefinite* if q_A has both positive and negative values.

Lemma

If D is a diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Then q_D is:

- positive definite if $\lambda_i > 0$ for $i = 1, \dots, n$
- positive semidefinite if $\lambda_i \geq 0$ for $i = 1, \dots, n$
- negative definite if $\lambda_i < 0$ for $i = 1, \dots, n$
- negative semidefinite if $\lambda_i \leq 0$ for $i = 1, \dots, n$
- indefinite if $\lambda_i > 0$ and $\lambda_j < 0$ for some i, j.

Proof: This is clear since

$$q_D\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2$$

Lemma

Let A be a symmetric matrix with an orthogonal diagonalization

$$A = QDQ^T$$

If the quadratic form q_D is positive definite (positive semidefinite etc.) then q_A has the same property.

Proof: (For positive semidefinite, other cases are similar)

If $q_D(v) \geq 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$

then $q_A(v) = q_D(Q^T v) \geq 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$.

Proposition

Let A be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ then the quadratic form q_A is

- positive definite if $\lambda_i > 0$ for $i = 1, \ldots, n$
- positive semidefinite if $\lambda_i \geq 0$ for $i = 1, \ldots, n$
- negative definite if $\lambda_i < 0$ for $i = 1, \ldots, n$
- negative semidefinite if $\lambda_i \leq 0$ for $i = 1, \ldots, n$
- indefinite if $\lambda_i > 0$ and $\lambda_j < 0$ for some i, j.

Example. Classify the quadratic form q_A defined by the matrix

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

Solution:

Characteristic polynomial of A

$$P(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 5 \\ 1 & 5-\lambda & 1 \\ 5 & 1 & 1-\lambda \end{bmatrix} = -\lambda^3 + 7\lambda^2 + 16\lambda + 112$$

$$= (7-\lambda)(4-\lambda)(-4-\lambda)$$

(Eigenvalues of A) = (roots of $P(\lambda)$) = $\{\lambda_1 = 7, \lambda_2 = 4, \lambda_3 = -4\}$

Thus q_A is indefinite.

Proposition

Let A be a symmetric matrix. The quadratic form q_A id positive semidefinite if and only if there exists a matrix B such that $A = B^T B$.

Proof:

We have seen before: if $A = B^T B$ then all eigenvalues of A are ≥ 0, so q_A is positive semidefinite

Conversely, if q_A - positive semidefinite then

$$A = QDQ^T, \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad \lambda_i \geq 0$$

Define $\sqrt{D} = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix}$ and let $B = Q\sqrt{D}Q^T$

We have:

$$B^T B = (Q\sqrt{D}Q^T)^T (Q\sqrt{D}Q^T)$$

$$= Q\sqrt{D}^T Q^T \cdot Q\sqrt{D}Q^T$$

$$\underset{Q^TQ = I}{=} Q\sqrt{D}^T\sqrt{D}Q^T \underset{\sqrt{D}^T = \sqrt{D}}{=} Q\sqrt{D}\cdot\sqrt{D}\cdot Q^T$$

$$= QDQ^T = A$$

Constrained optimization of quadratic forms

Note:

$$q_A(c\mathbf{v}) = (c\mathbf{v})^T A (c\mathbf{v}) = c^2(\mathbf{v}^T A \mathbf{v}) = c^2 q_A(\mathbf{v})$$

Constrained Maximum Problem. Given a symmetric matrix A, find a vector $\mathbf{v} \in \mathbb{R}^n$ such that $||\mathbf{v}|| = 1$ and the value $q_A(\mathbf{v})$ is the largest possible.

Lemma

If D is a diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then the vector $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$ is a solution of the Constrained Maximum Problem. Also, $q_D(\mathbf{e}_!) = \lambda_1$

Proof: For any vector $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ we have

$$q_D(v) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2$$

This gives: $q_D(e_1) = \lambda_1 \cdot 1^2 + \lambda_2 0^2 + \ldots + \lambda_n \cdot 0^2 = \lambda_1$

Also, for any vector $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ such that $||v|| = 1$ we get:

$$\begin{aligned} q_D(v) &= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2 \\ &\underset{\lambda_1 \geq \lambda_i \text{ for } i \geq 1}{\geq} \lambda_1 x_1^2 + \lambda_1 x_2^2 + \ldots + \lambda_1 x_n^2 = \lambda_1 (x_1^2 + x_2^2 + \ldots + x_n^2) \\ &\underset{||v||=1}{=} \lambda_1 = q_D(e_1) \end{aligned}$$

Proposition

Let A be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. If $\mathbf{u}_1$ is an eigenvector corresponding to λ_1 such that $||\mathbf{u}_1|| = 1$ then $\mathbf{u}_1$ is a solution of the Constrained Maximum Problem and $q_A(\mathbf{u}_1) = \lambda_1$.

Proof: We have

$$A = QDQ^T$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$Q = [u_1 \ u_2 \cdots u_n]$$

For any $v \in \mathbb{R}^n$ we have $q_A(v) = q_D(Q^T v)$.

Note: $\|Q^T v\| = \sqrt{(Q^T v)^T (Q^T v)} = \sqrt{v^T Q Q^T v} = \sqrt{v^T v} = \|v\| = 1$

By Lemma, $q_D(w)$ has the maximal value λ_1 for $w = e_1$. Thus $q_A(v)$ has the maximal value λ_1 if $Q^T v = e_1$

Since we have:

$$[e_1 \ e_2 \cdots e_n] = I = Q^T Q = [Q^T u_1 \ Q^T u_2 \cdots Q^T u_n]$$

we obtain: $Q^T u_1 = e_1$.

Example.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

We have seen before that eigenvalues of A are

$$\lambda_1 = 7, \quad \lambda_2 = 4, \quad \lambda_3 = -4$$

Thus the constrained maximum value of q_A is $q_A(u_1) = 7$ where u_1 is an eigenvector corresponding to $\lambda_1 = 7$ such that $\|u_1\| = 7$

$$(\text{eigenspace of } \lambda_1 = 7) = \mathrm{Nul}(A - 7 \cdot I)$$

$$= \mathrm{Nul}\left(\begin{bmatrix} -6 & 1 & 5 \\ 1 & -2 & 1 \\ 5 & 1 & -6 \end{bmatrix}\right)$$

$$\left[\begin{array}{ccc|c} -6 & 1 & 5 & 0 \\ 1 & -2 & 1 & 0 \\ 5 & 1 & -6 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We obtain: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 7$

Since $\|v_1\| = \sqrt{3}$, so we can take $u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$