Recall:

Definition

A square matrix A is a *diagonalizable* if A is of the form

$$A = PDP^{-1}$$

where D is a diagonal matrix and P is an invertible matrix.

Diagonalization Theorem

1) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

2) In such case $A = PDP^{-1}$ where :

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

λ_1 = eigenvalue corresponding to $\mathbf{v}_1$
λ_2 = eigenvalue corresponding to $\mathbf{v}_2$
... ...
λ_n = eigenvalue corresponding to $\mathbf{v}_n$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

eigenvalues : $\lambda_1 = 1$, $\lambda_2 = 5$

basis of the eigenspace of $\lambda_1 = 1$: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis of the eigenspace of $\lambda_2 = 5$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Thus A is diagonalizable:

$A = PDP^{-1}$ where $P = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Note. Not every matrix is diagonalizable.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The characteristic polynomial of A:

$$P(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - 0$$

$$(\text{eigenvalues of } A) = (\text{roots of } P(\lambda)) = \{\lambda = 2\}$$

<u>Eigenspace for $\lambda = 2$</u> $= \operatorname{Nul}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

basis of the eigenspace $= \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

Thus A has only one linearly independent eigenvector – not enough for diagonalization.

Recall:

Definition

If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in $\mathbb{R}^n$ then the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is the number

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \ldots + a_n b_n$$

Note: $\underset{\text{dot product}}{u \cdot v} = \underset{\text{matrix multiplication}}{u^T v}$

$u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $v = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$u^T = [a_1 \ \ldots \ a_n]$

$u^T v = [a_1 \ \ldots \ a_n] \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \ldots + a_n b_n$

Definition

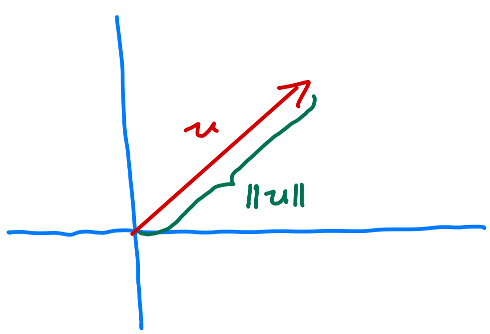
If $\mathbf{u} \in \mathbb{R}^n$ then the *length* (or the *norm*) of $\mathbf{u}$ is the number

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

$u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$u \cdot u = a_1^2 + \ldots + a_n^2$

$||u|| = \sqrt{a_1^2 + \ldots + a_n^2}$



Definition

Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

Definition

A square matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is an *orthogonal matrix* if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Example.

$$A = \begin{array}{c} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix} \\ \begin{array}{ccc} u_1 & \quad u_2 & \quad u_3 \end{array} \end{array}$$

$$u_1 \cdot u_1 = \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

$$u_1 \cdot u_2 = \frac{2}{3} \cdot \frac{1}{3} + \left(-\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 0$$

$\vdots$

Theorem

If Q is an orthogonal matrix then Q is invertible and $Q^{-1} = Q^T$.

Note:

$$A = \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} \end{array} \qquad A \cdot B = \begin{bmatrix} \overbrace{1 \cdot 7 + 2 \cdot 9}^{r_1 \cdot c_1} & \overbrace{1 \cdot 8 + 2 \cdot 0}^{r_1 \cdot c_2} \\ r_2 \cdot c_1 & r_2 \cdot c_2 \\ r_3 \cdot c_1 & r_3 \cdot c_2 \end{bmatrix}$$

$$Q = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & \cdots & u_n \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \qquad Q^T = \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots & u_1 \cdot u_n \\ u_2 \cdot u_1 & u_2 \cdot u_2 & \cdots & u_2 \cdot u_n \\ & & & \\ u_n \cdot u_1 & u_n \cdot u_2 & \cdots & u_n \cdot u_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

the identity matrix

Definition

A square matrix A is *symmetric* if $A^T = A$.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Spectral Theorem

If A is an $n \times n$ symmetric matrix then A has n orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

As a consequence the matrix A is diagonalizable:

$$A = QDQ^{-1} = QDQ^T$$

where

$$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_n \end{bmatrix}$$

is an orthogonal matrix with the orthonormal eigenvectors as columns and

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

is a diagonal matrix, where λ_i is an eigenvalue of A corresponding to the eigenvector $\mathbf{u}_i$.