Recall:

Definition

Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say that

- λ is an *eigenvalue* of A
- $\mathbf{v}$ is an *eigenvector* of A corresponding to λ.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad A \cdot v = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3v$

Thus v is an eigenvector of A corresponding to the eigenvalue $\lambda = 3$.

$w = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $\quad A \cdot w = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 1 \cdot w$

So w is an eigenvector of A corresponding to the eigenvalue $\lambda = 1$.

Note. Eigenvectors corresponding to a given eigenvalue λ form a subspace of $\mathbb{R}^n$ which is called the *eigenspace* corresponding to the eigenvalue λ.

Computation of eigenvalues

Notation. $I_n :=$ the $n \times n$ identity matrix.

Definition

If A is an $n \times n$ matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree n. $P(\lambda)$ is called the *characteristic polynomial* of the matrix A.

Proposition

If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$P(\lambda) = \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

$$(\text{Eigenvalues of } A) = (\text{roots of } P(\lambda)) = (\lambda_1 = 1,\ \lambda_2 = 5)$$

Computation of eigenvectors

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{\begin{array}{c}\text{eigenvectors of } A\\ \text{corresponding to } \lambda\end{array}\right\} = \left\{\begin{array}{c}\text{vectors in}\\ \mathrm{Nul}(A-\lambda I_n)\end{array}\right\}$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

From the previous example eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$

$$(\text{eigenspace of } \lambda_1 = 1) = \mathrm{Nul}(A - 1\cdot I) = \mathrm{Nul}\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - 1\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \mathrm{Nul}\left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right) = \left(\text{solutions of } \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\underbrace{\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \end{array}}_{\text{augmented matrix}} \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \underbrace{0}_{\text{free}} & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We obtain

$$\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \left(\begin{array}{c}\text{a basis}\\ \text{of } \mathrm{Nul}(A - 1\cdot I)\end{array}\right) = \left(\begin{array}{c}\text{a basis of}\\ \text{the eigenspace of } A\\ \text{corresponding to } \lambda_1 = 1\end{array}\right)$$