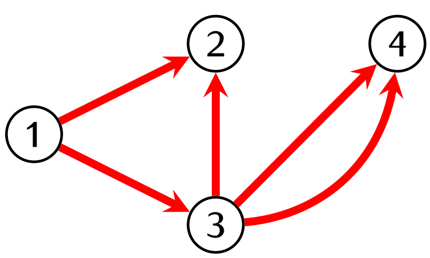
Definition

For a graph with vertices $1, 2, 3, \ldots, N$ the *adjacency matrix* of the graph is an $N \times N$ matrix $A = (a_{ij})$ such that

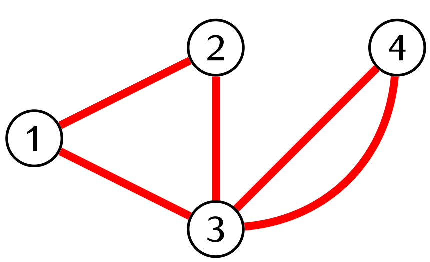
$$a_{ij} = (\text{the number of edges from } j \text{ to } i)$$

Example. Directed graph:



$$\begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 2 & 0\end{bmatrix}\end{array}$$

Example. Undirected graph:

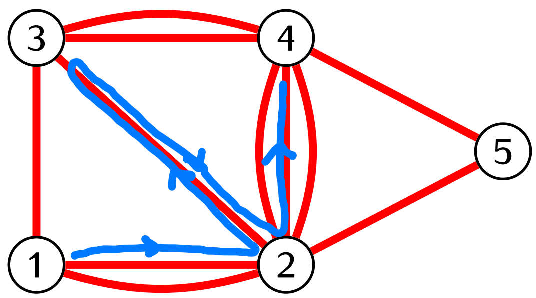


$$\begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 2\\ 0 & 0 & 2 & 0\end{bmatrix}\end{array}$$

<u>Note</u>: The adjacency matrix of an undirected graph is symmetric: $A = A^T$.

Definition

A *path* in a graph is a sequence of edges such that each edge ends at the vertex when the next edge begins.



Example. In the graph pictured above, how many paths of length 2 are there that start at the vertex 2 and end at the vertex 4?

Adjacency matrix:

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 3 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 3 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

← edges to vertex 4

↑ edges from vertex 2

$$\underbrace{\begin{array}{c c c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} & \\ 4 & \begin{bmatrix} 0 & 3 & 2 & 0 & 1 \end{bmatrix} & \cdot \end{array} \begin{array}{c c} 2 & \\ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}}_{} = \underbrace{0 \cdot 2}_{4 \leftarrow 1 \leftarrow 4} + \underbrace{3 \cdot 0}_{4 \leftarrow 2 \leftarrow 2} + \underbrace{2 \cdot 1}_{4 \leftarrow 3 \leftarrow 2} + \underbrace{0 \cdot 3}_{4 \leftarrow 4 \leftarrow 2} + \underbrace{1 \cdot 1}_{4 \leftarrow 5 \leftarrow 2}$$

↑ the entry in the 4^{th} row, 2^{nd} column of the matrix $A \cdot A$

Proposition

Let A be the the adjacency matrix of a graph.

The entry b_{ij} of the matrix $A^2 = (b_{ij})$ gives the number of paths of length 2 that start at the vertex j and terminate at the vertex i.

In general, for any $n \geq 1$ the entry c_{ij} of the matrix $A^n = (c_{ij})$ gives the number of paths of length n that start at the vertex j and terminate at the vertex i.