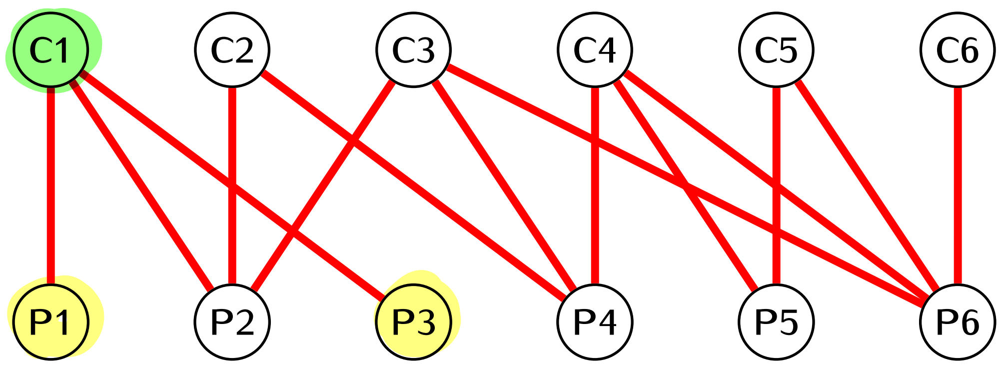
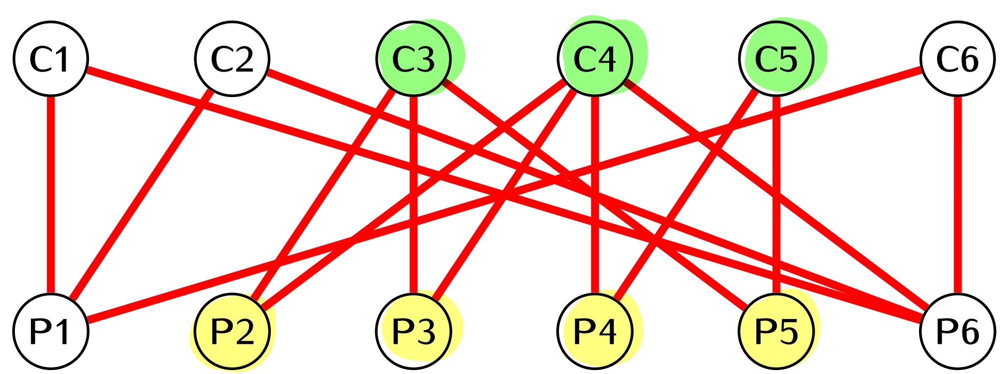
Example.



No solutions: There is only one candidate C1 that matches two positions P1 and P3.

Example.



No solutions: There are only 3 candidates matching the 4 positions P2, P3, P4, P5.

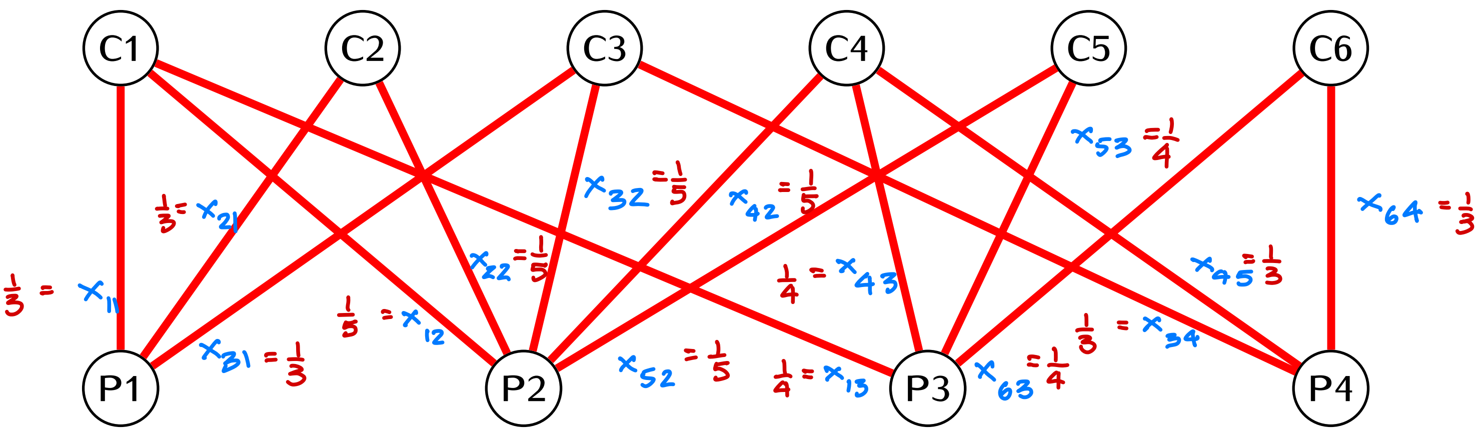
König's Theorem

Consider an assignment problem matching job candidates $C_1, \ldots, C_n$ with positions $P_1, \ldots, P_m$. Assume that there exists a number $k > 0$ such that

- for each $i = 1, \ldots, m$ there are at least k candidates who applied for the position P_i
- each candidate C_j applied for at most k positions.

Then the assignment problem has a solution. That is, it is possible to match each position with a job candidate, in such way that every position is filled and each job candidate has at most one position.

Proof. $k = 3$:



- We want to solve an integer program:

maximize $z = \sum_{ij} x_{ij}$

constraints: $\sum_j x_{ij} \leq 1$ (each candidate can get at most one position)

$\sum_i x_{ij} = 1$ (there is exactly one candidate selected for each position)

$0 \leq x_{ij} \leq 1$

$x_{ij} \in \mathbb{Z}$

- We had: it suffices to solve the relaxed linear program ($x_{ij} \in \mathbb{R}$).
- The linear program has a feasible solution:

For each j let $\deg(j)$ = number of edges adjacent to C_j

Then set : $x_{ij} = \frac{1}{\deg(j)}$

<u>Note</u>: $x_{ij} \leq \frac{1}{k}$.