

Definition

A matrix A is *totally unimodular* if every square matrix obtained by removing some rows and columns of A has determinant 0, 1, or -1 .

Note: 1) If A is totally unimodular then all entries of A are equal to 0, 1, or -1 .

2) Not every matrix with entries 1, 0, -1 is totally unimodular.

e.g.: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \det A = 2$

Proposition

Consider a linear program of the equality form: maximize

$$Z = c_1x_1 + \dots + c_nx_n$$

subject to constraints:

$$a_{i1}x_1 + \dots + a_{in}x_n = b_i$$

for $i = 1, \dots, m$ and $x_j \geq 0$ for $j = 1, \dots, n$.

If the coefficient matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

is totally unimodular and $b_i \in \mathbb{Z}$ for $i = 1, \dots, m$ then values of x_1, \dots, x_n for every basic feasible solution of the linear program are integers. In particular, if the linear program has a maximum, then there is a solution with integer values which gives this maximum.

Proof of Proposition.

Denote $A = [v_1 \ v_2 \ \dots \ v_n]$. Without loss of generality assume that we have a basic feasible solution where x_1, \dots, x_m are basic variables and x_{m+1}, \dots, x_n are free. Then we have $x_{m+1} = \dots = x_n = 0$, so we only need to show that x_1, \dots, x_m are integers.

We have:

$$A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{= b}$

so: $v_1 x_1 + v_2 x_2 + \dots + v_m x_m + \underbrace{v_{m+1} x_{m+1} + \dots + v_n x_n}_{= 0} = b$

Thus $x_1 v_1 + \dots + x_m v_m = b$

Let A_B be the $m \times m$ matrix with columns v_1, \dots, v_m :

$$A_B = [v_1 \ v_2 \ \dots \ v_m]$$

This gives: $A_B \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = b$

We have:

1) since x_1, \dots, x_m are basic variables, the columns v_1, \dots, v_m are linearly independent. This means that A_B is an invertible matrix. Thus $\det A_B \neq 0$.

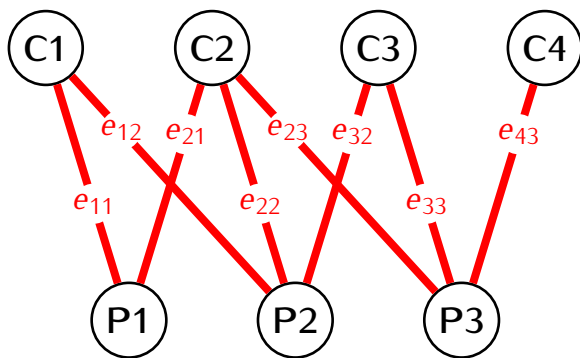
2) since A is totally unimodular we must have $\det A_B = \pm 1$ and all entries of A_B are integers.

This gives that x_1, \dots, x_m are integers.

Proposition

For any bipartite graph $G = (V_1 \cup V_2, E)$ the incidence matrix of G is totally unimodular.

Example.



	e_{11}	e_{12}	e_{21}	e_{22}	e_{23}	e_{32}	e_{33}	e_{43}
C1	1	1	0	0	0	0	0	0
C2	0	0	1	1	1	0	0	0
C3	0	0	0	0	0	1	1	0
C4	0	0	0	0	0	0	0	1
P1	1	0	1	0	0	0	0	0
P2	0	1	0	1	0	1	0	0
P3	0	0	0	0	1	0	1	1

A

Induction:

Assume that all submatrices of A of size $n \times n$ have determinant 0, 1 or -1.

Let B - submatrix of A of size $(n+1) \times (n+1)$

Consider three cases:

1) B has a column consisting of zeros.
Then $\det B = 0$ - ok

2) B has a column that contains only one entry equal to 1.

Then by cofactor expansion with respect to this column we get $\det B = \pm \det(B')$

where B' is an $n \times n$ matrix, so $\det B' = 0, 1$ or -1 .

Thus $\det B = 0, 1$ or -1 .

3) Every column of B contains two entries equal to 1

Then rows of B are linearly dependent:

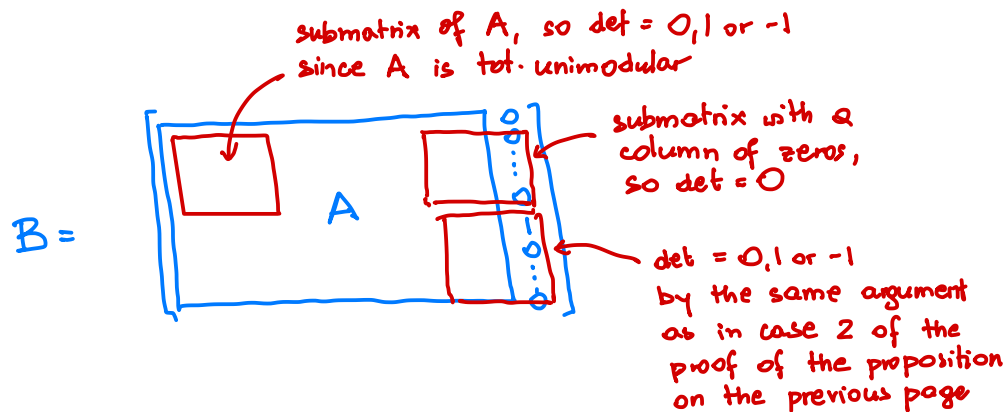
(sum of rows corresp. to C_i 's) - (sum of rows corresp. to P_i 's) = 0

Thus B is not an invertible matrix, and $\det B = 0$ - ok.



Proposition

If A is a totally unimodular matrix and B is a matrix obtained by appending to A by a column that has only one non-zero entry equal to 1, then B is totally unimodular.



Corollary

The simplex method always gives a solution to the assignment problem that consists of integers.

Proof: The constraints of the assignment problem are of the form $Ax = b$ where A is totally unimodular matrix and $b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Thus all basic feasible solutions of this problem consist of integers.